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Factorization conjecture and the open/closed string correspondence

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Abstract: We present evidence for the factorization of the world-sheet path integrals for 2d conformal field theories on the disk into bulk and boundary contributions. This factorization is then used to reinterpret a shift in closed string backgrounds in terms of boundary deformations in background independent open string field theory. We give a proof of the factorization conjecture in the cases where the background is represented by WZW and related models.

Keywords: String Field Theory, Conformal Field Models in String Theory, D-branes.

Contents

1. Introduction

Since the early days of string theory it has been suspected that the distinction between open strings and closed strings should not be fundamental. This follows already from the observation that closed string poles occur as intermediate states in open string scattering amplitudes. From the point of view of open string field theory these poles seem to violate unitarity unless closed string states are present in the classical open string field theory. One possibility is to accept that open string field theory is not unitary and to add extra closed string degrees of freedom by hand [1]. However, in this approach one has to address the problem of overcounting since now the same diagram can be obtained from the open and closed string sector of the field theory lagrangian.

An alternative approach is to try to identify closed string states directly in open string field theory [2, 3]. This idea receives further motivation from Sen's work on non-BPS branes [4, 5] which resulted in a very active study of open string field theory in different formulations and some progress in understanding the vacuum structure of open strings has been achieved (see [6] for a review).

In this paper we suggest an approach based on the idea that closed string degrees of freedom correspond to (non-local) boundary interactions advocated in [7, 8] within the framework of background independent open string field theory (BSFT) [7, 9−15]. BSFT is defined by the path integral over σ-model fields for a fixed closed string background $\mathcal{X}$ with the dynamical open string degrees of freedom t corresponding to boundary deformations of the CFT on the disk. It is, however, important to note that these deformations are not required to be local on the boundary of the world-sheet [9−12]. Once non-local boundary perturbations are included the distinction between open and closed degrees of freedom on a world-sheet with boundary becomes ambiguous. In fact in the early days of background independent open string theory it was realized that the notion of locality on the world-sheet was a major question to be addressed since deformations on the boundary were described by a limiting procedure of taking the closed string operator from the bulk and moving it to the boundary. The simplest way to identify $\mathcal{X}$ is by means of the closed string σ-model lagrangian. $\mathcal{X}$ then defines a conformal σ-model background in the absence of boundaries. In the examples studied in this paper we will make some natural choices in this regard and then demonstrate a relation between them.

The key ingredient in our approach is the factorization property for the BSFT space-time action $\mathcal{S}(\mathcal{X}|t)$,

$$\mathcal{S}(\mathcal{X}|t) = \mathcal{Z}_0(\mathcal{X})\mathcal{S}_0(\mathcal{X}|t)\,. \tag{1.1}$$

Here $\mathcal{Z}_0(\mathcal{X})$ is the D-instanton partition function and $\mathcal{S}_0(\mathcal{X}|t)$ is described purely in terms of the quantum mechanical degrees of freedom Φ_b on the boundary. Given the relation between the world-sheet partition function and the BSFT action (see equation (2.1) below) this property is a consequence of the following conjecture for 2d conformal field theories on a manifold with boundary:

$$\int_{\Phi|_{\partial\Sigma}=\Phi_b(\theta)} D[\Phi]\, e^{\frac{i}{\alpha'}I_\Sigma(\Phi)} = \mathcal{Z}_0(\mathcal{X})\, e^{\frac{i}{\alpha'}I^{bdry}(\Phi_b)}\,. \tag{1.2}$$

Here $I_\Sigma(\Phi)$ is a lagrangian for the world-sheet conformal field theory on a 2d surface with boundary, and $\mathcal{Z}_0(\mathcal{X})$ is the D-instanton partition function which is given by (1.2) for $\Phi_b = 0$. We verify this property the case of when Σ is a disk, in the situation where ghosts and matter decouple and for $\mathcal{X}$ such that the closed string world-sheet is conformal and described in terms of a WZW (or related) model. These technical assumptions are necessary since not much is known about BSFT when ghosts and matter do not decouple.

The logic underlying our approach is the following: To each 2d CFT with boundary corresponds a boundary action $I^{bdry}(\Phi_b)$. Due to the factorization property, $I^{bdry}(\Phi_b)$ is independent of α' but certainly depends on the CFT chosen on the left hand side of eq. (1.2). The ambiguity in this process is under control (see section 3). On the other hand, in the reconstruction of the bulk CFT for a given boundary action there may be further ambiguities. We then claim that there is a distinction between the class of bulk theories reconstructed from boundary actions I^{bdry}, differing by (non-local) functionals of the boundary field Φ_b.

Note that the boundary action plays a central role in BSFT since one integrates over all maps from the world-sheet to the target space without specifying the boundary conditions. One starts from a boundary action and considers the class of its boundary deformations; this class contains all other boundary actions with the same number of boundary fields Φ_b (or less). The boundary actions corresponding to boundary conformal field theories on the world-sheet are, by definition of the string field theory action, solutions of the classical equations of motion for $\mathcal{S}(\mathcal{X}|t)$. These are in turn critical points in t for fixed $\mathcal{X}$ and denoted by t_*. The space-time action $\mathcal{S}(\mathcal{X}|t_* + t_q)$ expanded around t_* to n-th order in t_q is supposed to reproduce the n-point open string amplitudes for the background defined by t_*. This is known to be true on classical level in the space-time field theory corresponding to disk amplitudes on the world-sheet.

Concretely we start with a closed string background $\mathcal{X}$ and find t^1_*. Then we look for a second critical point of $\mathcal{S}_0$: t^2_*; since $\mathcal{X}$ is a "hidden variable" in the open string field theory action $\mathcal{S}_0$ we need to reconstruct it for the new critical point t^2_*. This in general is a difficult problem and in principle might be ambiguous. Even so we can argue that in the set of critical points of $\mathcal{S}_0(\mathcal{X}|t)$ there are critical points $t = t^1_*$ and $t = t^2_*$ such that the expansion around t^2_* is identical to the expansion around t^1_* but for different closed string background $\mathcal{X}'$, i.e.

$$\mathcal{S}_0(\mathcal{X}|t^1_* + t_q) = \mathcal{S}_0(\mathcal{X}'|t^2_* + t_q)\,. \tag{1.3}$$

Thus, a deformation from t^1_* to t^2_* can be interpreted as deforming the closed string background from $\mathcal{X}$ to $\mathcal{X}'$.

A simple realization of the conjectured property (1.3) leads to the Seiberg-Witten map [16]: It is well known that a constant Kalb-Ramond B-field can be seen equivalently as a closed string background $\mathcal{X}$ or a perturbation on the boundary of the open string world-sheet, i.e. $\mathcal{S}_0(\mathcal{X} = (G, B)|t_q) = \mathcal{S}_0(\mathcal{X}' = (G, 0)|t_* + t_q)$. The result of [16] can then be formulated as the statement that the expansion around t_* leads to non-commutative field theory in Minkowski space. The generalization to a non-constant, closed B-field leads to Kontsevich's deformation quantization [17]. At present we allow for arbitrary B compatible with bulk conformal invariance.

Note that the factorization of the world-sheet partition function into bulk and boundary contributions is crucial for the closed string degrees of freedom to be contained in open string field theory. Indeed if bulk α'-corrections entered in the definition of $I^{bdry}(\Phi_b)$ one would get different α'-expansions for the open and closed string β-function. The factorization property, which guarantees that closed string fluctuations do not feed back into the definition of the open string field theory, is instrumental for the open-closed string correspondence to work. This appears to be a very subtle distinction between bulk conformal field theories in 2d and general 2d QFT where this factorization does not hold in general.

As a warm up and to fix the notation, we apply our formalism in section 2 to radius deformations on the torus where the decoupling is immediate. As an example for a curved closed string background we then prove the factorization property for boundary WZW models with arbitrary boundary conditions to all orders in perturbation theory in section 3. This requires a definition of WZW models with boundary conditions which are not of the class $J = R\bar{J}$ [18–22], rather only implying $T - \bar{T} = \beta^i(t)V_i(t) = 0$, where $V_i(t)$ is a boundary perturbation and $\beta^i(t)$ its β-function. Finally we illustrate our result considering the SU(2)-WZW model in the large radius limit in section 4.

2. BSFT on a Torus

In the case when ghost and matter fields decouple the definition of the space-time action in flat space is written in terms of the disk partition function $\mathcal{Z}(t)$ and boundary β-function

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as [12][1]

$$\mathcal{S}(\mathcal{X}|t) = \left(1 - \beta^i \frac{\partial}{\partial t^i}\right) \mathcal{Z}(\mathcal{X}|t) \,. \tag{2.1}$$

Here t^i are the couplings representing the open string degrees of freedom and β^i denotes the β-function associated to the coupling t^i.

For our purpose we suggest a different normalization of the space-time action by replacing $\mathcal{Z}(\mathcal{X}|t)$ by $\mathcal{Z}^{bdry}(\mathcal{X}|t) \equiv \mathcal{Z}(\mathcal{X}|t)/\mathcal{Z}_0(\mathcal{X})$, where $\mathcal{Z}_0(\mathcal{X})$ is the "D-instanton" partition function, which is independent of the open string background $\{t\}$. Of course this normalization assumes the factorization of the CFT on the disk, which we will prove shortly. Our normalization does not alter the dynamics of the open string fields t^i, therefore we can work with $\mathcal{S}_0(\mathcal{X}|t)$ instead of $\mathcal{S}(\mathcal{X}|t)$,

$$\mathcal{S}_0(\mathcal{X}|t) = \left(1 - \beta^i \frac{\partial}{\partial t^i}\right) \mathcal{Z}^{bdry}(\mathcal{X}|t) \,. \tag{2.2}$$

To start with we consider the free action for maps X from the disk into a circle of radius R

$$S(X) = \frac{R^2}{4\pi i \alpha'} \int_D \partial X \bar{\partial} X \,, \tag{2.3}$$

where $\partial \equiv dz\partial_z$. The radius R plays the role of a closed string modulus. According to BSFT we are instructed to integrate over maps with free boundary conditions, which leads to the notion of the boundary field f defined through $X(z,\bar{z})|_{\partial D} = f(\theta)$; boundary deformations are functionals of f, in general non-local. This field f can be unambiguously extended from the boundary to the interior of the disk via harmonicity condition (harmonic functions are solutions of the world-sheet equations of motion). Every field $X(z,\bar{z})$ may thus be split into a harmonic boundary field and a bulk field which obeys Dirichlet conditions,

$$X(z,\bar{z}) = X_0(z,\bar{z}) + X_b(f) \,, \tag{2.4}$$

such that $X_0|_{\partial D} = 0$ and $X_b(z,\bar{z})|_{\partial D} = f(\theta)$ with $\Delta X_b = 0$, so $X_b(f)$ is a harmonic function with value $f(\theta)$ on the boundary.

Note that the boundary field can always be expanded as $f = \sum_n f_n e^{in\theta}$, which suggests a separation into chiral and anti-chiral modes corresponding to positive and negative frequencies. Thus, $f = f^+ + f^- + f_0$ can then be extended to $X_b(f) = f^+(z) + f^-(\bar{z}) + f_0$. Moreover there is a reality condition $f^{+*} = f^-$. The zero mode f_0 plays the role of the space-time integration variable in the space-time action.

Plugging this ansatz into the free action (2.3) the mixed terms containing X_0 and X_b vanish after partial integration. The action splits into

$$S(X) = \frac{R^2}{4\pi i \alpha'} \int \partial X_0 \bar{\partial} X_0 + \frac{R^2}{4\pi i \alpha'} \int \partial f^+ \bar{\partial} f^- \,. \tag{2.5}$$

[1] Note that this expression is written without use of a metric on the space of boundary interactions, only the vector field β^i is required.

Given the translation invariance of the measure in this example the factorization property is obviously satisfied. The partition function then reads

$$\mathcal{Z}(R) = \mathcal{Z}_0(R) \int D[f]\, e^{-\frac{R^2}{4\pi i\alpha'} \int \partial f^+ \bar\partial f^-}\,, \tag{2.6}$$

where

$$\mathcal{Z}_0(R) = \int D[X_0]\, e^{-\frac{R^2}{4\pi i\alpha'} \int \partial X_0 \bar\partial X_0}\,, \tag{2.7}$$

supplemented by the b, c ghost system is the "D-instanton" partition function.[2] Since $f^\pm$ is harmonic its contribution takes the form of a non-local boundary interaction

$$I^{bdry}(f) = \frac{R^2}{4\pi} \oint f H(f) = \frac{R^2}{4\pi} \oint\oint d\theta d\theta' f(\theta) H(\theta,\theta') f(\theta') = \frac{R^2}{2\pi} \oint f^+ \partial_\theta f^-\,, \tag{2.8}$$

where H is a Hilbert transform $H(f) = \partial_n f = \partial(f_- - f_+)$ and $H(\theta,\theta') = \frac{1}{4\pi i} \sum_n e^{in(\theta-\theta')}|n|$. Integration over f with this boundary interaction then produces the partition function of a D-brane extended along the X-direction.

To be more general we can add local interactions on the boundary, parametrized by couplings $\{t_q\}$. They are given by functionals of f, so that the local and non-local contributions can be collected into

$$\begin{aligned} I^{bdry}(t, X) &= I^{bdry}(t, f)\,, & (2.9)\\ \mathcal{Z}^{bdry}(R|t) &= \int D[f] e^{\frac{i}{\alpha'} I^{bdry}(t,f)}\,. & (2.10) \end{aligned}$$

From the above it is now clear that a change in the closed string modulus $R \to R + \delta R$ appears as a deformation of the boundary interaction

$$\begin{aligned} I^{bdry} &\to I^{bdry} + \delta I^{bdry} \\ \delta I^{bdry}(R) &= \frac{R\delta R}{2\pi} \oint f H(f)\,. \end{aligned} \tag{2.11}$$

In the presence of open string degrees of freedom this is a non-trivial modification of the boundary theory. For instance for the Euclidean D1-brane wrapping S^1 the condition for marginality of the boundary operator $\exp ikX_b$ with $k = n \in \mathbb{Z}$ is changed by the shift[3] $R \to R + \delta R$. We thus conclude that the modulus R of the closed string background $\mathcal{X} = S^1_R$ enters as a non-local boundary interaction. In particular,

$$\mathcal{S}_0(\mathcal{X} = S^1_{R+\delta R} | t^1_* + t_q) = \mathcal{S}_0(\mathcal{X} = S^1_R | t^2_* + t_q)\,, \tag{2.12}$$

in accord with (1.3). Note that the theory without additional boundary interactions is conformally invariant for any R. Therefore there is no β-function associated to the radius deformation. But the β-functions for other couplings depend on the non-local part (and therefore on the bulk moduli) of the boundary interaction.

After this warm-up we will now consider interacting CFTs. In the next section we show that the factorization property also holds for boundary conformal theories on group manifolds.

[2] Here we take the conventional boundary conditions for b and c, because decoupling of matter and ghost sector is assumed.

[3] Similarly, strings attached to the D-instanton can wind around S^1. Their contribution to the boundary partition function is represented by the insertion of boundary vertex operators $\exp i \frac{wR}{\alpha'} X_b$.

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3. Boundary WZW model

The prototype example for open strings propagating in curved space-time is the WZW model which is also an example where the B-field is not closed. Here we will discuss this case in detail. Other curved target spaces can be treated in a similar fashion.

As is well known [18 – 20] in this case world-sheets Σ with boundary $\partial\Sigma$ require some care in the definition of the topological term $\Gamma(g) = \int_\Gamma \mathrm{tr}(dgg^{-1})^3$ with $\partial\Gamma = M$. For a closed 2d surface M this term is defined as an integral of a 3-form over a 3-manifold Γ with the 2d surface M as its boundary. If the 2d surface has a boundary the unambiguous definition of this term is problematic.

We need the condition $H^3(G) = 0$ on the group G in order to define the topological term in the WZW model in terms of a globally well-defined 2-form w_2 such that $dw_2(g) = w_3(g) = \mathrm{tr}(dgg^{-1})^3$ (since w_3 is a closed 3-form, $dw_3 = 0$, such w_2 always exists locally). We write this formally as $w_2(g) = d^{-1}w_3(g)$. If $H^3(G) = \mathbb{Z}$ there is no such globally well-defined w_2, but $\Gamma(g) = \int_M w_2$ is still globally well-defined modulo $\mathbb{Z}$ as long as M has no boundaries. If $M = \Sigma$ has a boundary, one needs the condition $H^3(G) = 0$ in order to define $\int_\Sigma w_2(g)$ for an arbitrary map $g : \Sigma \to G$. This is the case, for instance, for $\mathrm{SL}(2,\mathbb{R})$ which we will now consider. However, even in this case $\Gamma(g)$ is not unique since any w_2' that differs from w_2 by an exact 2-form,

$$w_2' = w_2(g) + d\beta(g)\,, \tag{3.1}$$

leads to the same w_3. In general $d\beta$ is closed but not necessarily exact. Thus, the action $\Gamma(g)$ is defined by the 3-form w_3 up to an ambiguity that comes from the 1-form β, which contributes to the action only through a boundary term

$$\Gamma^\beta(g) = \int_\Sigma w_2(g) + \int_{\partial\Sigma} \beta(g^b)\,. \tag{3.2}$$

We denote by g^b the restriction of g to the boundary. If β is not well-defined globally, $\int_\Sigma d\beta$ still makes sense and depends only on g^b since for two different continuations of g^b into the bulk the difference is $\int_{S^2} d\beta = 0 \bmod \mathbb{Z}$.

For $\mathrm{SL}(2,\mathbb{R})$, (3.2) can serve as definition of a class of WZW actions together with the standard kinetic term

$$I_{\mathrm{WZW}} = \frac{\kappa}{4\pi i}\int_\Sigma \mathrm{tr}\,(\partial_\mu g g^{-1})^2 + \frac{\kappa}{4\pi i}\Gamma^\beta(g)\,. \tag{3.3}$$

One expects the theory to be exactly conformally invariant for particular choices of the boundary term $\int_{\partial\Sigma}\beta$. Classifying such 1-forms β is an interesting question, in particular, in view of solutions to the quantum conformality condition $T = \bar{T}$ on the boundary which do not reduce to the condition that g^b belongs to a fixed conjugacy class, which in turn follows from the equations for the currents $J = \bar{J}$ on the boundary. The latter constraint is, in fact, stronger than the conformality condition.

Let us now see how the procedure described for free scalar field in the previous section is modified in this case. From $dw_2 = w_3$ it follows immediately on the level of differential forms that

$$\gamma(g_1, g_2) \equiv w_2(g_1g_2) - w_2(g_1) - w_2(g_2) + \mathrm{tr}\; g_1^{-1}dg_1dg_2g_2^{-1}\,. \tag{3.4}$$

is a well-defined closed 2-form. We note in passing that (3.4) is closed without restriction to $H^3(G) = 0$. Furthermore, γ defines a 2-cocycle on the loop group $\hat{LG}$. Indeed, if we integrate the closed 2-form (3.4) over the disk with boundary S^1, we get $\alpha_2(g_1^b, g_2^b) = \int_D \gamma(g_1, g_2)$, where g^b is the restriction of g to the boundary and this α_2 satisfies the cocycle condition. To see that α_2 only depends on the boundary data of g_1 and g_2, we note that for two different extension g_i^+ and g_i^- of g_i^b the difference

$$\int_{D^+} \gamma - \int_{D^-} \gamma = \int_{S^2} \gamma = 0 \bmod \mathbb{Z} \tag{3.5}$$

as a consequence of (3.4). Since g_i^+ and g_i^- are the same on the boundary and are otherwise independent, the result follows. The fact that α_2 satisfies cocycle condition can be checked by direct algebraic computation using (3.4) (see also [23 − 26]).

To continue we will use the following decomposition (motivated by the free field example in the previous section) for a generic map from the disk Σ to the group G:

$$g(z, \bar{z}) = g_0(z, \bar{z})k(z, \bar{z}); \qquad g_0|_{\partial\Sigma} = 1; \qquad k|_{\partial\Sigma} = f(\theta)\,, \tag{3.6}$$

so g_0 describes the D-instanton and k is purely defined by the boundary data $f(\theta) : S^1 \to G$. We will give a concrete definition of k below. For $H^3(G) = 0$ each 2-form appearing on the rhs of (3.4) is separately well-defined, so that

$$\int_\Sigma w_2(g_0k) = \int_\Sigma w_2(g_0) + \int_\Sigma w_2(k) - \int_\Sigma \mathrm{tr}\; g_0^{-1}dg_0dkk^{-1} \tag{3.7}$$

mod$\mathbb{Z}$ since the 2-cocycle $\alpha_2(g_1^b \equiv 1, g_2^b \equiv f) = 0 \bmod \mathbb{Z}$ [27]. Combined with the kinetic term in (3.2) this leads to the expression

$$I_{\mathrm{WZW}}(g) = I_{\mathrm{WZW}}(g_0) + \frac{\kappa}{4\pi i}\int_\Sigma \mathrm{tr}\,(\partial_\mu kk^{-1})^2 + \frac{\kappa}{4\pi i}\Gamma^\beta(k) + \frac{\kappa}{2\pi i}\int_\Sigma \mathrm{tr}\; g_0^{-1}\bar{\partial}g_0\partial kk^{-1}\,. \tag{3.8}$$

This action is well-defined though the theory depends on the 1-form β through the boundary integral $\int_{S^1}\beta(f)$ in $\Gamma^\beta(k)$.

In order to proceed we will now specify the extension $k(z, \bar{z})$ of the boundary data $f(\theta)$ by solving the Riemann-Hilbert problem for $f(\theta)$. This means we decompose $f(\theta)$ as

$$f(\theta) = h_+(\theta)h_-(\theta)\,, \tag{3.9}$$

where h_+ can be holomorphically continued to $h(z)$ into the disk Σ and h_- anti-holomorphically to $\bar{h}(\bar{z})$. Thus, we have for $k(z, \bar{z})$

$$k(z, \bar{z}) = h(z)\bar{h}(\bar{z})\,. \tag{3.10}$$

Here h and $\bar{h}$ are fields on the complexified group.[4] This $k(z, \bar{z})$ solves the WZW equations of motion and, together with g_0, gives an unique decomposition of an arbitrary field $g = g_0k$ on the disk. We will take (3.8) with this decomposition as definition of the WZW model

[4]The factors h and $\bar{h}$ can be constructed by solving the equation of motion in Minkowski signature, that is, $k(\sigma^+, \sigma^-) = h(\sigma^+)\bar{h}(\sigma^-)$ where h and $\bar{h}$ are independent functions and then define $h(z)$ and $\bar{h}(\bar{z})$ by analytic continuation.

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on the disk for arbitrary boundary fields $f(\theta)$ taking values in the group manifold. In background independent open string field theory we are instructed to integrate over $f(\theta)$. As we emphasized above this WZW theory on the disk depends on the 1-form β on the boundary, and since this 1-form is completely arbitrary we include it in the definition of the boundary perturbation in BSFT. We do not specify for which β this theory is conformal — this is a good question and the only comment we will make is that the string field theory action is one candidate for the solution — its critical points correspond to conformal boundary interactions parametrized by $\beta(f)$. We conclude that the WZW theory on the disk for the case $H^3(G) = 0$ is given by the action (3.8) with the definitions (3.9), (3.10) and (3.6).

3.1 Bulk-boundary factorization

Unlike for the free field case, in the classical WZW action (3.8) the boundary field k does not decouple from the bulk fields g_0 on the level of the classical action. The interaction between these two fields is given by

$$\frac{\kappa}{2\pi i} \int_\Sigma d^2 z \bar{J}_{g_0} \partial K \, , \tag{3.11}$$

where $\bar{J}_{g_0}$ is the anti-holomorphic g_0-current, and the holomorphic function $K(z)$ is defined via $\partial K = \partial k k^{-1}$ using the fact that $\partial k k^{-1}$ is a holomorphic 1-form.

Nevertheless, we will show below that this cross term in (3.8) between g_0 and f (which parametrizes k) does not contribute to the path integral over g_0. Concretely we will prove that the n-point function

$$\left\langle \left(\int_\Sigma \mathrm{tr}\; g_0^{-1} \bar{\partial} g_0 \partial k k^{-1} \right)^n \right\rangle_{g_0} = 0 \, . \tag{3.12}$$

Thus for any choice of β

$$\int_{g|_{\partial\Sigma}=f} D[g] e^{-I_{\mathrm{WZW}}(g)} = \mathcal{Z}_0 e^{-W(f)} \, , \tag{3.13}$$

where

$$W(f) = \frac{\kappa}{4\pi i} \int_\Sigma \mathrm{tr}\, (\partial_\mu k k^{-1})^2 + \frac{\kappa}{4\pi i} \Gamma^\beta(k) \tag{3.14}$$

and

$$\mathcal{Z}_0 = \int_{g_0|_{\partial\Sigma}=1} D[g_0] e^{-I_{\mathrm{WZW}}(g_0)} \, , \tag{3.15}$$

verifying our conjectured factorization in this class of models. This is the main technical result of this paper

We will now give a qualitative argument for the vanishing of the n-point function (3.12). The explicit proof of this claim is given in the appendix. Consider the functional integral over g_0 at fixed k. This is the WZW theory with boundary conditions in the identity conjugacy class. The correlators of $\bar{J}$'s are functions with poles in $\bar{z}$, but no positive powers of $\bar{z}$ occur. These correlators are then multiplied by functions ∂K which are polynomials of positive powers of z. These products are proportional to a positive power of $e^{i\theta}$ so that the θ integral vanishes as long as no singularities occur and the $U(1)$-action by $e^{i\theta}$ is unbroken. In the appendix we show that no such singularities appear.[5] A similar situation appears e.g. in Kontsevich's work [17] in proving the vanishing theorem for deformation quantization.

The crucial property for these arguments to work is that the n-point functions of the antichiral currents $\bar{J}$ on the disk are functions of $\bar{z}$ only. In general one might expect interactions of the currents with their images. This would generate terms which behave singular at the boundary. But in this particular case no such terms appear, and this is due to the following argument: It is important that in this n-point function only chiral bulk fields are involved in the WZW theory with $g_0 = 1$ at the boundary (for the trivial conjugacy class), i.e. we are interested in the expectation values $\langle \bar{J}(\bar{z}_1) \cdots \bar{J}(\bar{z}_p) \rangle_D$ with Dirichlet boundary conditions. This amplitude has an equivalent representation in terms of a Dirichlet boundary state $|B_D\rangle$. The explicit construction of $|B_D\rangle$ is not needed. We merely need to assume that such a state exists. Then the expectation value can be written as an unnormalized correlation function $\langle 0|\bar{J}(\bar{z}_1) \cdots \bar{J}(\bar{z}_p)|B_D\rangle$. Expanding the normal ordered product of currents in modes we get

$$\sum_{n_1 \cdots n_p} \bar{z}_1^{n_1} \cdots \bar{z}_p^{n_p} \langle 0| : \bar{j}_{n_1} \cdots \bar{j}_{n_p} : |B_D\rangle \, . \tag{3.16}$$

All $\bar{j}_n$ with $n \leq 0$ annihilate on the vacuum, thus the expression contains only terms with $n > 0$. The boundary state is defined by $\bar{J} d\bar{z}|B_D\rangle = J dz|B_D\rangle$, thus it maps $\bar{j}_n$ to j_{-n}. Since the holomorphic and anti-holomorphic currents commute, the j_{-n} can be moved all the way to the left to act on the vacuum, which it annihilates. This then implies that bulk normal ordered monomials of chiral operators have a vanishing expectation value also for Dirichlet boundary conditions. For the ordinary product of antiholomorphic currents we then conclude that

$$\langle 0|\bar{J}(\bar{z}_1) \cdots \bar{J}(\bar{z}_p)|B_D\rangle \propto \langle 0|\bar{J}(\bar{z}_1) \cdots \bar{J}(\bar{z}_p)|0\rangle \, , \tag{3.17}$$

that is, the boundary state enters only in the normalization. Thus the only singularities are those of coinciding $\bar{J}$'s, which can then be treated in the manner described above.

To summarize, this line of argument shows that, although the bulk and boundary fields to not decouple in the classical action, the partition function is independent of the interaction term $\int \bar{J}_{g_0} \partial K$ to any order in perturbation theory. Thus the boundary degrees of freedom decouple from the bulk and the partition function factorizes. To complete the argument we note that the translation invariance of the functional Haar measure $D[g]$ implies that no jacobian occurs when integrating out the bulk fields g_0. Thus

$$\int_{g|_{\partial\Sigma}=f} D[g] e^{-I_{\mathrm{WZW}}(g)} = \mathcal{Z}_0 e^{-W(f)} \, , \tag{3.18}$$

[5] It should be noted that this argument works only because we integrate over the disk. One-dimensional integrals of such perturbations over the boundary of the disk would give rise to divergences [28].

where

$$W(f) = \frac{\kappa}{4\pi i}\int_\Sigma \mathrm{tr}\,(\partial_\mu k k^{-1})^2 + \frac{\kappa}{4\pi i}\Gamma^\beta(k)\,. \tag{3.19}$$

An immediate consequence of the above result is that the boundary partition function on a group manifold is related to the flat space partition function by a non-local boundary deformation in agreement with the correspondence stated in the introduction.

3.2 Groups with $H^3(G) \neq 0$

Let us now turn to the case when $H^3(G) = \mathbb{Z}$ such as $G = \mathrm{SU}(N)$. In this case w_2 is not globally defined (it is ill-defined on a high codimension submanifold of the target, which is just a point for the case of the $\mathrm{SU}(2) \cong S^3$ group manifold). Let us follow the arguments for the $H^3(G) = 0$ case and see where the problems show up. In order to reduce the action to the form (3.8) we use the decomposition $g = g_0 k$, with k as in (3.9). Since (3.4) is still well-defined we can formally arrive at the equation (3.7). In particular, the non-trivial 2-cocycle $\alpha_2(g_0^b, f)$ after formula (3.7) is again zero for $g_0^b = 1$. There is no problem to globally define the first and the last term on the rhs of (3.7). The difficulty resides in the second term $w_2(k)$. Thus the problem is with the definition of the WZW action on solutions of the classical equations of motion $\bar\partial(\partial k k^{-1}) = 0$, with $k|_{\partial\Sigma} = f(\theta)$, where f is arbitrary. In this case the classical lagrangian turnes out to be not a function anymore [29, 30]. However, this might be expected, because a path integral with boundary conditions defines a wave-function, which corresponds to a section of some bundle. Note that although the action is ambiguous the equations of motion derived from it are well-defined.

Recall that the reason we want to consider boundary conditions which are not in a conjugacy class is that according to the philosophy of BSFT one has to integrate over all degrees of freedom including the boundary fields with boundary interactions parametrized by the 1-form β. From the expression (2.1) for the string field theory action, it follows that it is the space-time action (2.1) that needs to be well-defined for boundary deformations and not the world-sheet classical action $W(f) = \frac{\kappa}{4\pi i}\int \mathrm{tr}\,(\partial_\mu k k^{-1})^2 + \frac{\kappa}{4\pi i}\Gamma^\beta(k)$. That is, an integral over boundary maps

$$\mathcal{Z}^{bdry} = \mathcal{Z}/\mathcal{Z}_0 = \int D[f] e^{-W(f)}\,, \tag{3.20}$$

shall be well-defined, where $D[f]$ is a Haar measure for k written in terms of f after expressing k via the solution of the Riemann-Hilbert problem described above. Even if this path integral diverges ultimately, it is the combination entering in (2.1) that shall lead to a well-defined space-time action.

Since there are infinitely many choices for β one would like to classify them according the conformality condition for the corresponding quantum theory. As we mentioned for $\mathrm{SL}(2,\mathbb{R})$, this is exactly the question that background independent open string field theory studies.

For $H^3(G) \neq 0$, one way to remove the topological obstruction in defining $\Gamma^\beta(k)$, is by deleting a high codimension submanifold in G and repeating (3.4) for $g_1 = g_0$ and $g_2 = k$. Since these relations are algebraic we still can safely derive the formal relation (3.7). One might then suggest that in this case a $d\beta$ can be found, so that the integral over boundary fields is still well-defined as mentioned above (with appropriate regularization procedure). We recall that a similar situation appears for the analogous quantum-mechanical problem for trajectories with boundaries in a compact phase-space (associated with coadjoint orbits, and related), where the classical action on the world-line is ill-defined due to non-trivial H^2 of the phase space though the path integral can be properly defined in order to get a correct wave-function[6] [31]. In short — although the action is ill-defined on high codimension submanifolds the path integral on the manifold with boundary still gives a well-defined and correct "wave-function" (matrix element). According to [32, 33] our current problem is an infinite-dimensional version of the quantum mechanical problem. We believe that the same is true for the family of 2d field theories related to WZW models on the disk for group manifolds with non-trivial H^3.

Critical points of the string field theory action (2.1) are supposed to lead to well-defined conformal boundary conditions and well-defined $\mathcal{Z}^{bdry} = \mathcal{Z}/\mathcal{Z}_0$, which is the value of the space-time action on-shell according to (2.1) (these boundary interactions, in particular, do contain the restriction to conjugacy classes as a sub-set of the conformal conditions).

So at the moment we simply assume that (3.3) be given via (3.8) for all groups including those with $H^3(G) \neq 0$ (as we mentioned for $\mathrm{SL}(2,\mathbb{R})$, everything is properly defined in (3.8) and this case is very intersting on its own right) and define the string field theory action via standard methods.

At this point a comment about the measure $D[k]$ is in order. If we start with the Haar measure for g, the natural measure for k comes out to be the functional Haar measure for $k(z,\bar{z}) = h(z)\bar{h}(\bar{z})$. Note, however, that k is uniquely determined in terms of the boundary data. When pulling back $D[k]$ to the boundary a jacobian occurs and introduces a further non-locality in the boundary interaction. So the total non-local boundary deformation resulting from a shift in the closed string background is given by $W(f) = \frac{\kappa}{4\pi i}\int_\Sigma \mathrm{tr}\,(\partial_\mu k k^{-1})^2 + \frac{\kappa}{4\pi i}\Gamma^\beta(k)$ plus the jacobian generated. In the next section we will give an illustration by considering the large radius limit of the SU(2) model.

4. Illustrative example and final remarks

As we have already mentioned there is no unique way to fix β since each choice is related to a choice of boundary interactions. In principle β's corresponding to conformal open string backgrounds can be constructed order by order in perturbation theory imposing scale invariance at each order. As starting point we take the action

$$S_\Sigma(g_0 k) = S_\Sigma(g_0) + \frac{\kappa}{2\pi i}\int_\Sigma \mathrm{tr}\ g_0^{-1}\bar{\partial} g_0 \partial k k^{-1} + \frac{\kappa}{4\pi i}\int \mathrm{tr}\ k^{-1}\partial k k^{-1}\bar{\partial} k\,. \qquad (4.1)$$

In fact the last term can be written in terms of the boundary data $f(\theta)$ (via its decomposition (3.9)) by using the fact that h and $\bar{h}$ are holomorphic and thus writing

[6] Here, w_3 is the Kirillov-Kostant symplectic 2-form. For a given w_2 (which is a 1-form then) an appropriate $d\beta$ (which also is a 1-form then) can be found such that the path integral for an open trajectory, where the boundaries are points now, gives a matrix element in an irreducible representation of the compact group.

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$h^{-1}\partial h = \partial K^{+}(f)$ and $\bar{\partial}\bar{h}\bar{h}^{-1} = \bar{\partial}K^{-}(f)$ This leads to an expression in terms of the Hilbert transform $\oint d\theta K H(K)$, where $K = K^{+} + K^{-}$ (see (2.8)). This form makes the dependence on f more explicite. In the "abelian limit" the latter becomes exactly the formula for the boundary action we derived for free fields.

We now take (4.1) as action for an SU(2) WZW model and consider the first order perturbation in $1/\sqrt{\kappa}$. That is we use the parametrization $k = \exp i \frac{X_b^i \sigma^i}{\sqrt{\kappa}}$, where σ^i are the Pauli matrices. These coordinates have a simple interpretation in the large-κ-limit. They become the usual flat coordinates and the boundary action (last term in (4.1)) leads upon integration over the boundary fields to a space-filling brane (Neumann boundary conditions in all directions). However, we do not expect this boundary interaction to be conformal for finite κ. This is because the quantum mechanical propagator $\frac{1}{\partial_n}$ of the boundary theory is not standard and needs to be renormalized. The boundary fields X_b can be re-expressed in terms of holomorphic and antiholomorphic continuations of the boundary data $f^i(\theta)$ to $f^i(z)$ and $\bar{f}^i(z)$,

$$X_b^i = f^i + \bar{f}^i - \frac{1}{\sqrt{\kappa}} \epsilon^i_{jk} f^j \bar{f}^k \,. \tag{4.2}$$

Here the large-κ-limit has been taken, including only the first correction. Writing the boundary interaction in terms of these fields we get

$$W(f) \sim \oint \left\{ f^i \bar{\partial} \bar{f}_i + \frac{1}{\sqrt{\kappa}} \epsilon^{ijk} f_i \bar{f}_j \partial f_k - \frac{1}{\sqrt{\kappa}} \epsilon^{ijk} f_i \bar{f}_j \bar{\partial} \bar{f}_k \right\}, \tag{4.3}$$

which is a non-local function of the flat space boundary fields $f^i(z,\bar{z}) = f^i(z) + \bar{f}^i(\bar{z})$. Next we consider the jacobian from expressing the boundary Haar measure D[k] in terms of the flat space measure. To first order the correction to the measure is then given by the formal expression

$$-\frac{1}{\sqrt{\kappa}} \operatorname{tr} \frac{\delta}{(\delta f^l, \delta \bar{f}^l)} \epsilon^{ijk} \left[f^j \bar{f}^k \right]^{\pm} , \tag{4.4}$$

where $\pm$ stands for decomposition in holomorphic and anti-holomorphic modes. Variation will always produce factors δ^{jl} or δ^{kl}. As the trace also includes a contraction of the indices i and l, the first correction to the measure vanishes. Therefore the first correction to the boundary partition function is

$$\mathcal{Z}^{bdry}\left(t^i\right) = \left\langle 1 - \frac{1}{\sqrt{\kappa}} \oint \epsilon^{ijk} f_i \bar{f}_j \partial f_k + \frac{1}{\sqrt{\kappa}} \oint \epsilon^{ijk} f_i \bar{f}_j \bar{\partial} \bar{f}_k \right\rangle_{I^{bdry}(t^i)} . \tag{4.5}$$

The expectation value is taken with respect to the canonical action in flat space $\frac{1}{2} \oint f \partial_n f$ and a boundary interaction I^{bdry}, using the flat measure. This expression can now be used to determine the first order contribution of β by imposing scale invariance.

To summarize, in this paper we have given evidence for the conjecture that within the framework of BSFT different closed string σ-model backgrounds can be equivalently described in terms of non-local open string backgrounds. This serves as a test for the idea that closed string degrees of freedom are indeed contained in the classical open string field theory.

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A. Proof of bulk/boundary factorization

In this appendix we give an explicit proof of the claim in section 3.1 that that (3.11) does not contribute to the path integral over the bulk field g_0. We choose coordinates $z = \rho e^{i\theta}$ on the disk ($|z| \leq 1$). The operator $\exp - \int \bar{J}_{g_0} \partial K$ is expanded as $\sum (n!)^{-1}(-1)^n I_n$, where

$$\begin{aligned} I_n &\equiv \int d^2 z_1 \partial_1 K(z_1) \cdots \int d^2 z_n \partial_n K(z_n) \mathcal{A}_n(\bar{z}_1, \ldots, \bar{z}_n) \\ \mathcal{A}_n &\equiv \left\langle \bar{J}_{g_0}(\bar{z}_1) \cdots \bar{J}_{g_0}(\bar{z}_n) \right\rangle . \end{aligned} \tag{A.1}$$

Here $J_{g_0} = g_0^{-1} \bar{\partial} g_0$ is the anti-holomorphic bulk current. The basic ingredient for computing the integral (A.1) is the OPE of the anti-holomorphic currents

$$\bar{J}^a(\bar{z}_1) \bar{J}^b(\bar{z}_2) \sim \frac{\kappa \delta^{ab}}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{i f^{abc}}{\bar{z}_1 - \bar{z}_2} \bar{J}^c(\bar{z}_2) , \tag{A.2}$$

where $\bar{J} = \kappa^{-1} \bar{J}^a T^a$, T^a are the generators of the algebra, f^{abc} the structure constants and δ^{ab} the Cartan metric. But we will see that the calculation does not depend on details like symmetry structures of the group.

As general strategy we evaluate the indefinite integrals in order to treat the singularities correctly. The result is then shown to be a regular function of all variables, so that the boundaries can be inserted and no singularities occur.

It is clear that the one-point function vanishes, I_1=0. The two-point function is more involved since the Wick theorem does not hold and there are self-interactions of the currents. The amplitude is

$$\mathcal{A}_2(\bar{z}_1, \bar{z}_2) = \left\langle \frac{k \delta^{ab}}{(\bar{z}_1 - \bar{z}_2)^2} + \frac{i f^{abc}}{\bar{z}_1 - \bar{z}_2} \bar{J}^c(\bar{z}_2) \right\rangle T^a T^b \propto \frac{1}{(\bar{z}_1 - \bar{z}_2)^2} . \tag{A.3}$$

We expand the holomorphic field as

$$\partial K(z) = \sum_{m>0} m K_m z^{m-1} . \tag{A.4}$$

Thus, I_2 consists of (a sum of) terms

$$m_1 m_2 \oint d\theta_1 e^{i(m_1-1)\theta_1} \oint d\theta_2 e^{i(m_2-1)\theta_2} \int_0^1 d\rho_1 \int_0^1 d\rho_2 \frac{\rho_1^{m_1} \rho_2^{m_2} e^{2i\theta_1}}{(\rho_1 - \rho_2 e^{-i(\theta_2 - \theta_1)})^2} . \tag{A.5}$$

The structure becomes more obvious when a relative boundary coordinate $\theta = \theta_2 - \theta_1$ is introduced,

$$m_1 m_2 \oint d\theta_1 e^{i(m_1+m_2)\theta_1} \oint d\theta e^{i(m_2-1)\theta} \int_0^1 d\rho_1 \int_0^1 d\rho_2 \frac{\rho_1^{m_1}\rho_2^{m_2}}{(\rho_1 - \rho_2 e^{-i\theta})^2} . \tag{A.6}$$

As $m_i \geq 1$ the θ_1-integral makes the whole term vanish as long as the remaining integrals are not divergent. The m_i are set to 1, because higher powers of ρ_i will, at best, smoothen the singularities. We conduct the ρ_1-integral and the relevant part becomes

$$\int d\theta \int d\rho_2 \left[\rho_2 \ln(\rho_1 - \rho_2 e^{-i\theta}) - \frac{\rho_2^2 e^{-i\theta}}{\rho_1 - \rho_2 e^{-i\theta}}\right] . \tag{A.7}$$

The second part of (A.7) is

$$\begin{aligned} &\int d\theta e^{-i\theta}\left[\frac{1}{2}\rho_2^2 e^{i\theta} + \rho_1\rho_2 e^{2i\theta} + \rho_1^2 e^{3i\theta}\ln(\rho_1 - \rho_2 e^{-i\theta})\right] = \\ &= \rho_1^2 \int d\theta e^{2i\theta} \ln\left(\rho_1 - \rho_2 e^{-i\theta}\right) + \text{regular terms} \end{aligned} \tag{A.8}$$

Conducting the θ-integral yields

$$-\frac{i}{2}\left(\rho_1^2 e^{2i\theta} - \rho_2^2\right) \ln\left(\rho_1 e^{i\theta} - \rho_2\right) + \text{regular terms}\,, \tag{A.9}$$

which is non-singular in all variables. Therefore the whole expression is non-divergent and vanishes finally under the θ_1-integral.

The first part of (A.7) is, after ρ_2-integration,

$$\frac{1}{2}\int d\theta \left[\rho_2^2 \ln\left(\rho_1 - \rho_2 e^{-i\theta}\right) + e^{-i\theta}\int d\rho_2 \frac{\rho_2^2}{\rho_1 - \rho_2 e^{-i\theta}}\right] . \tag{A.10}$$

The whole expression becomes, using the result from (A.8),

$$-\frac{1}{2}\int d\theta e^{2i\theta}\left(\rho_1^2 - \rho_2^2 e^{-2i\theta}\right) \ln\left(\rho_1 - \rho_2 e^{-i\theta}\right) + \text{regular terms}\,. \tag{A.11}$$

This term is regular even without θ-integration. Therefore all terms are finite and finally vanish under the θ_1-integral. Thus $I_2 = 0$.

The three-point-amplitude is proportional to

$$\mathcal{A}_3(\bar{z}_1, \bar{z}_2, \bar{z}_3) \propto \frac{1}{(\bar{z}_1 - \bar{z}_2)(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_3)} . \tag{A.12}$$

I_3 contains terms of the form

$$\int d\rho_1 d\theta_1 \cdots d\rho_3 d\theta_3 \frac{m_1 m_2 m_3 \rho_1\rho_2\rho_3}{(\rho_1 e^{-i\theta_1} - \rho_2 e^{-i\theta_2})(\rho_2 e^{-i\theta_2} - \rho_3 e^{-i\theta_3})(\rho_1 e^{-i\theta_1} - \rho_3 e^{-i\theta_3})} . \tag{A.13}$$

Again we set $m_i = 1$ in order to single out the most singular part. The indefinite integration over ρ_1 gives

$$\begin{aligned} &\oint d\theta_1 \int d\rho_1 \frac{\rho_1\rho_2\rho_3}{(\rho_1 e^{-i\theta_1} - \rho_2 e^{-i\theta_2})(\rho_2 e^{-i\theta_2} - \rho_3 e^{-i\theta_3})(\rho_1 e^{-i\theta_1} - \rho_3 e^{-i\theta_3})} = \\ &= \frac{\rho_2^2 e^{-i\theta_2}\rho_3}{(\rho_2 e^{-i\theta_2} - \rho_3 e^{-i\theta_3})^2} \oint d\theta_1 e^{2i\theta_1} \ln\left(\rho_1 e^{-i\theta_1} - \rho_2 e^{-i\theta_2}\right) - \left[\rho_2 e^{-i\theta_2} \leftrightarrow \rho_3 e^{-i\theta_3}\right] . \end{aligned} \tag{A.14}$$

Now we conduct the θ_1-integral[7]

$$\int d\theta_1 e^{i\theta_1} \ln\left(\rho_1 e^{-i\theta_1} - \rho_2 e^{-i\theta_2}\right)$$
$$= i\rho_1 \left[\frac{\rho_1 e^{-i\theta_1} - \rho_2 e^{-i\theta_2}}{\rho_1 e^{-i\theta_1}\rho_2 e^{-i\theta_2}} \ln\left(\rho_1 e^{-i\theta_1} - \rho_2 e^{-i\theta_2}\right) - \frac{e^{i\theta_2}}{\rho_2} \ln\left(\rho_1 e^{-i\theta_1}\right)\right]. \quad \text{(A.15)}$$

Restoring the pre-factors from (A.14) we see that (A.15) is less singular than

$$\frac{i\rho_2\rho_3 e^{i\theta_1}}{\bar{z}_{23}^2}\left(\bar{z}_{12}\ln\bar{z}_{12} - \bar{z}_1\ln\bar{z}_1\right) - \left[\rho_2 e^{-i\theta_2} \leftrightarrow \rho_3 e^{-i\theta_3}\right]. \quad \text{(A.16)}$$

The expression in the bracket is completely regular. As pre-factor we recognize the contribution from the 2-point function. Thus, we conclude that I_3 must have the same or a less singular behavior than I_2. Thus, the overall θ_1-integration, which is also present for the three-point function, makes the whole expression vanish, $I_3 = 0$.

This argument can now be applied recursively to n-point functions. For the sake of a clear presentation we switch to a rather symbolic notation. The recursion then works like (modulo some permutations)

$$\int dz_n \frac{(\cdots)}{(\bar{z}_1 - \bar{z}_2)(\bar{z}_2 - \bar{z}_3)\cdots(\bar{z}_{n-1} - \bar{z}_n)(\bar{z}_n - \bar{z}_1)} \propto \frac{(\cdots)}{(\bar{z}_1 - \bar{z}_2)(\bar{z}_2 - \bar{z}_3)\cdots(\bar{z}_{n-1} - \bar{z}_1)} + \\ + \text{less singular terms} \quad \text{(A.17)}$$

until one ends up with a three-point amplitude. Thus, all these indefinite integrals are indeed regular.

Now we argue that in fact all the integrals I_n must vanish. We extract $e^{i\theta_1}$ from each factor $(\bar{z}_i - \bar{z}_j)^{-1}$ and shift all the other boundary coordinates $\theta_i \to \theta_i' = \theta_i - \theta_1$. This gives a global factor of $\exp i\sum_i m_i\theta_1$. The m_i are always positive, thus the θ_1-integration makes the whole expression vanish. We arrive at the central result of this calculation:

$$I_n = 0\,. \quad \text{(A.18)}$$

The immediate consequence is that the operator $\exp - \int \bar{J}_{g_0}\partial K$ is marginal and therefore the partition function does not depend on it.

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